

EVENTUALLY LINEAR PARTIALLY COMPLETE RESOLUTIONS OVER A LOCAL RING WITH $\mathfrak{m}^4 = 0$

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ABSTRACT. We classify the Hilbert polynomial of a local ring (R, \mathfrak{m}) satisfying $\mathfrak{m}^4 = 0$ which admits an eventually linear resolution \mathbf{C} which is ‘partially’ complete — that is, for which $H^i \operatorname{Hom}_R(\mathbf{C}, R)$ vanishes for all $i \gg 0$. As a corollary to our main result, we show that an $\mathfrak{m}^4 = 0$ local ring can admit certain classes of asymmetric partially complete resolutions only if its Hilbert polynomial is symmetric. Moreover, we show that the Betti sequence associated to an eventually linear partially complete resolution over an $\mathfrak{m}^4 = 0$ local ring cannot be periodic of period two or three.

INTRODUCTION

Let (R, \mathfrak{m}) be a commutative local Noetherian ring with unique maximal ideal \mathfrak{m} . A finitely generated R -module M is said to be totally reflexive if it is reflexive and if both $\operatorname{Ext}_R^i(M, R)$ and $\operatorname{Ext}_R^i(\operatorname{Hom}_R(M, R), R)$ vanish for all $i > 0$. It is straightforward to see that every free R -module is totally reflexive, but not *vice versa*. For example, over a Gorenstein local ring, the maximal Cohen-Macaulay modules and totally reflexive modules coincide. Furthermore, while there are several known examples of non-free totally reflexive modules over non-Gorenstein rings (cf. [3, 5, 15, 16], among others), the classification of all local rings which admit such modules is far from complete.

Perhaps the structurally simplest non-Gorenstein local rings (R, \mathfrak{m}) to admit non-free totally reflexive modules are those satisfying the condition $\mathfrak{m}^3 = 0$ ($\neq \mathfrak{m}^2$). In 2003, Yuji Yoshino provided a characterization of such rings; cf. [17, Theorem 3.1]. In particular, the author proved that if a local ring (R, \mathfrak{m}) satisfying $\mathfrak{m}^3 = 0$ admits a non-free totally reflexive module M , then (a) the Betti numbers of M are constant, (b) R is Koszul, and (c) the Hilbert polynomial of R is balanced — that is, it has a root of -1 .

One can see that Yoshino’s characterization in [17] does not extend to the class of local rings (R, \mathfrak{m}) satisfying $\mathfrak{m}^4 = 0$ ($\neq \mathfrak{m}^3$); indeed, there are many examples of totally reflexive modules with non-constant Betti numbers over such rings. Even more noteworthy, however, is an example constructed by Jorgensen and Şega in 2005, in which the authors exhibit a local ring (R, \mathfrak{m}) satisfying $\mathfrak{m}^4 = 0$ and a totally reflexive R -module M such that the Betti numbers of M are constant, while the Betti numbers of $\operatorname{Hom}_R(M, R)$ grow exponentially; [9, Theorem 1.2].

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The results in this paper are motivated by those of both [17] and [9], but with respect to a more general class of modules: those with eventually linear partially complete resolutions, where the latter characteristic is defined by the vanishing of $\text{Ext}_R^i(M, R)$ and $\text{Ext}_R^i(\text{Hom}_R(M, R), R)$ for all $i \gg 0$. Given a local ring (R, \mathfrak{m}) satisfying $\mathfrak{m}^4 = 0$ which admits such a module M , we would like to (a) characterize the Hilbert polynomial of R , and (b) give necessary conditions for M to possess an asymmetric (partially) complete resolution. However, many of our results concerning the Hilbert polynomial of R regard a still more general class of modules: those which satisfy at least the vanishing of $\text{Ext}_R^i(M, R)$ for $i \gg 0$.

In Sections 1 and 2 we outline preliminary definitions and concepts which are extensively used throughout the paper. Section 3 is concerned with the growth of the Betti sequence; in addition to identifying particular growth rates, this section builds up to the following result.

Theorem A. *Let M be a finitely generated module with an eventually linear minimal free resolution over a local ring (R, \mathfrak{m}) satisfying $\mathfrak{m}^4 = 0$. If $\text{Ext}_R^*(M, R)$ eventually vanishes, then the Betti sequence of M is not eventually periodic of period two or three.*

The main goal of Section 4 is to characterize the Hilbert polynomial of an $\mathfrak{m}^4 = 0$ local ring admitting eventually linear partially complete resolutions with certain types of acyclicity. However, as mentioned above, most of our results — including the following — are stated in a more general setting.

Theorem B. *Let M be a finitely generated module with an eventually linear minimal free resolution over a local ring (R, \mathfrak{m}) satisfying $\mathfrak{m}^4 = 0$, and suppose that $\text{Ext}_R^*(M, R)$ eventually vanishes.*

- (1) *If the Betti sequence of M has non-exceptional polynomial growth, then the Hilbert polynomial of R is symmetric.*
- (2) *If the Betti sequence of M has exponential growth of base a , then the Hilbert polynomial of R takes the form $H_R(t) = 1 + et + ft^2 + gt^3$, where*

$$f = \left(a + \frac{1}{a}\right)e - \left(a^2 + 1 + \frac{1}{a^2}\right) \quad \text{and} \quad g = e - \left(a + \frac{1}{a}\right).$$

In particular, we use the above result to prove, in Theorem 4.3.1, that the type of acyclic complete resolution exhibited in [9] can only be admitted by a ring with a symmetric Hilbert polynomial. We further prove, in Theorem 4.3.3, that certain exponential vs. exponential acyclic complete resolutions do not exist over $\mathfrak{m}^4 = 0$ local rings.

1. PRELIMINARY CONCEPTS

Unless otherwise stated, R shall represent the commutative local (Noetherian) ring (R, \mathfrak{m}, k) having unique maximal ideal \mathfrak{m} and residue class field $k := R/\mathfrak{m}$. Furthermore, whenever a local ring satisfies the condition $\mathfrak{m}^4 = 0$, it shall also be assumed that $\mathfrak{m}^3 \neq 0$.

1.1. Total reflexivity. Although the concept of total reflexivity can be defined more generally over an arbitrary Noetherian ring, cf. [4, Section 2], our results are concerned with its existence over a certain class of local rings.

Definition 1.1.1. A finitely generated R -module M is said to be *totally reflexive* if each of the following conditions hold.

- (1) The canonical map $M \rightarrow \text{Hom}_R(\text{Hom}_R(M, R), R)$ is an isomorphism.
- (2) $\text{Ext}_R^i(M, R) = 0$ for all $i > 0$.
- (3) $\text{Ext}_R^i(\text{Hom}_R(M, R), R) = 0$ for all $i > 0$.

We denote by M^* the ring dual $\text{Hom}_R(M, R)$. Notice that condition (1) above yields the classical reflexivity condition $M \cong M^{**}$.

Remark 1.1.2. There exist finitely generated reflexive modules which are not totally reflexive; for example, if $R = k[[x_1, x_2, \dots, x_n]]$, then $\Omega_2(k)$ fits the bill. Still, the overall independence of conditions (1)–(3) in Definition 1.1.1 is not completely understood over an arbitrary local ring. In [10], however, Jorgensen and Şega demonstrate the independence of conditions (2) and (3) for a finitely generated reflexive R -module M .

Definition 1.1.3. Given a totally reflexive R -module M , let $\mathbf{F} = (F_i, \partial_i)$ and $\mathbf{G} = (G_i, d_i)$ be minimal free resolutions of M and M^* , respectively. Then the totally acyclic complex defined by

$$\begin{array}{ccccccc} \cdots & \longrightarrow & F_2 & \xrightarrow{\partial_2} & F_1 & \xrightarrow{\partial_1} & F_0 \xrightarrow{\delta} G_0^* \xrightarrow{d_1^*} G_1^* \xrightarrow{d_2^*} G_2^* \longrightarrow \cdots \\ & & & & \searrow \pi & \nearrow \iota & \\ & & & & & M & \end{array}$$

such that $\delta := \iota \circ \pi$, is called the *complete resolution* of M over R , and is often denoted $\mathbf{F}|\mathbf{G}^*$.

In fact, given any totally acyclic complex $\mathbf{C} = (C_i, \partial_i)_{i \in \mathbb{Z}}$ over R , the R -module coker ∂_j is totally reflexive for each $j \in \mathbb{Z}$.

1.2. Linear resolutions. Let $\mathbf{C} = (C_i, \partial_i)_{i \in \mathbb{Z}}$ be a minimal complex of R -modules. Then the *associated graded complex* of \mathbf{C} with respect to \mathfrak{m} is given by

$$\text{gr}_{\mathfrak{m}}(\mathbf{C}) := \bigoplus_{j \in \mathbb{Z}} \text{gr}_{\mathfrak{m}}(\mathbf{C})^j$$

where

$$\text{gr}_{\mathfrak{m}}(\mathbf{C})^j : \quad \cdots \longrightarrow \frac{\mathfrak{m}^{j-i-1}C_{i+1}}{\mathfrak{m}^{j-i}C_{i+1}} \xrightarrow{\delta_{i+1}^j} \frac{\mathfrak{m}^{j-i}C_i}{\mathfrak{m}^{j-i+1}C_i} \xrightarrow{\delta_i^j} \frac{\mathfrak{m}^{j-i+1}C_{i-1}}{\mathfrak{m}^{j-i+2}C_{i-1}} \longrightarrow \cdots$$

such that, for each $d \in \mathbb{Z}$, the map δ_d^j is induced by the restriction of $\mathfrak{m}^{j-d}C_d \rightarrow \mathfrak{m}^{j-d+1}C_{d-1}$, modulo $\mathfrak{m}^{j-d+1}C_d$. We also maintain the convention that $\mathfrak{m}^n = R$ for $n \leq 0$.

Remark 1.2.1. The associated graded complex is often referred to as being the *linear part* of a minimal complex, precisely because it filters non-linear behavior from the differentials. As one would imagine, this sort of a construction does not always preserve the exactness of a complex.

Definitions 1.2.2. Let M be a finitely generated R -module with minimal free resolution $\mathbf{F} \twoheadrightarrow M$. If $\text{gr}_{\mathfrak{m}}(\mathbf{F})$ is exact in positive degrees, then \mathbf{F} is said to be a *linear resolution* of M . In addition, if M is totally reflexive and the minimal free resolution $\mathbf{G} \twoheadrightarrow M^*$ is linear, then $\mathbf{F}|\mathbf{G}^*$ is said to be a *linear complete resolution* of M .

In general, given a minimal free resolution $\mathbf{F} \twoheadrightarrow M$ over R , the quantity defined by

$$\mathrm{ld}_R(M) := \sup \{n \mid H_i(\mathrm{gr}_{\mathfrak{m}}(\mathbf{F})) = 0 \text{ for all } i \geq n\}$$

is called the *linearity defect* of M . If $\mathrm{ld}_R(M)$ is finite and positive, then \mathbf{F} is said to be *eventually linear*. If, given a totally reflexive module M , the quantities $\mathrm{ld}_R(M)$ and $\mathrm{ld}_R(M^*)$ are both finite and positive, then the complete resolution of M is called eventually linear.

Remark 1.2.3. Simply stated, a minimal complex of free R -modules is (eventually) linear if and only if the nonzero entries of the matrices defining each (sufficiently high) differential in the complex are elements of $\mathfrak{m} \setminus \mathfrak{m}^2$.

Definition 1.2.4. If the minimal free resolution $\mathbf{F} \twoheadrightarrow M$ is linear, then M is said to be a *Koszul module*. Notice that if, instead, $\mathrm{ld}_R(M) = n < \infty$, then the n th syzygy module of M , denoted $\Omega_n(M)$, is Koszul. Furthermore, (R, \mathfrak{m}, k) is called a *Koszul ring* whenever k is Koszul as an R -module.

1.3. Hilbert series and Poincaré series. Let k be an arbitrary field, and suppose that $V = \bigoplus_{i \geq 0} V_i$ is a graded vector space over k such that $\dim_k V_n < \infty$ for each $n \in \mathbb{N}$. Then recall that the formal power series given by

$$H_V(t) := \sum_{i \geq 0} \dim_k V_i t^i \in \mathbb{Z}[[t]]$$

is called the *Hilbert series* of V . Thus, the Hilbert series of the associated graded ring of (R, \mathfrak{m}, k) is given by

$$H_{\mathrm{gr}_{\mathfrak{m}}(R)}(t) := \sum_{i \geq 0} \dim_k (\mathfrak{m}^i / \mathfrak{m}^{i+1}) t^i.$$

One should note that if $\mathfrak{m}^n = 0$ for some $n < \infty$, then the Hilbert series of $\mathrm{gr}_{\mathfrak{m}}(R)$ is simply a polynomial.

Furthermore, for any finitely generated R -module M , one can also speak of the Hilbert series of the associated graded module of M , which is defined by

$$H_{\mathrm{gr}_{\mathfrak{m}}(M)}(t) := \sum_{i \geq 0} \dim_k (\mathfrak{m}^i M / \mathfrak{m}^{i+1} M) t^i.$$

If $\mathbf{F} \twoheadrightarrow M$ is a minimal free resolution, then the quantity

$$\mathrm{rank} F_n := \dim_k \mathrm{Tor}_n^R(M, k)$$

is called the n th *Betti number* of M , and shall be denoted $b_n(M)$, or simply b_n when there is no risk of confusion as to the module. Often throughout this paper, we will refer to the *Betti sequence* of M over R , which is given by $\{b_i(M)\}_{i \geq 0}$.

Furthermore, the formal power series

$$P_M^R(t) := \sum_{i \geq 0} b_i(M) t^i \in \mathbb{Z}[[t]]$$

is called the *Poincaré series* of M over R .

Fact 1.3.1. *If (R, \mathfrak{m}, k) is Koszul, then the Poincaré series of k as an R -module is given by*

$$P_k^R(t) = \frac{1}{H_{\mathrm{gr}_{\mathfrak{m}}(R)}(-t)}$$

In the next section, we develop a system of equations which will be used to prove each of our main results.

2. THE SYSTEM OF EQUATIONS

The following lemma is a result of Herzog and Iyengar which characterizes, in terms of the ring alone, the Poincaré series of a finitely generated module with an eventually linear minimal free resolution. We state it without proof.

Lemma 2.1. [8, Proposition 1.8] *Let M be a finitely generated module over a local ring (R, \mathfrak{m}) . If the minimal free resolution of M is eventually linear, then*

$$P_M^R(t) = \frac{q(t)}{H_{\text{gr}_{\mathfrak{m}}(R)}(-t)(1+t)^{\dim R}}$$

for some $q \in \mathbb{Z}[t]$.

Note that if $\text{Ext}_R^i(M, R)$ vanishes for all $i \gg 0$, the above lemma will characterize the Poincaré series of a sufficiently high syzygy in the variable $\frac{1}{t}$. Furthermore assuming that R is zero-dimensional, one obtains the following result.

Proposition 2.2. *Let M be a finitely generated module over a zero-dimensional local ring (R, \mathfrak{m}) such that $\text{Ext}_R^i(M, R) = 0$ for all $i \gg 0$. If the minimal free resolution of M is eventually linear, then there exists $n \in \mathbb{N}$ such that the following hold.*

- (1) $P_{\Omega_n(M)}^R(t) H_{\text{gr}_{\mathfrak{m}}(R)}(-t) \in \mathbb{Z}[t]$
- (2) $P_{\Omega_n(M)}^R(\frac{1}{t}) H_{\text{gr}_{\mathfrak{m}}(R)}(-t) \in \mathbb{Z}[t]$

Proof. (1) follows from Lemma 2.1, a proof of which can be found in [8].

In order to prove (2), let $n \in \mathbb{N}$ be such that $\text{Ext}_R^i(M, R) = 0$ for $i > n$. Then, letting (R^{b_i}, ∂_i) denote the minimal free resolution of M , one has that the sequence

$$0 \rightarrow \Omega_n(M)^* \rightarrow R^{b_n} \xrightarrow{\partial_{n+1}^*} R^{b_{n+1}} \xrightarrow{\partial_{n+2}^*} R^{b_{n+2}} \xrightarrow{\partial_{n+3}^*} R^{b_{n+3}} \rightarrow \dots$$

is exact. The additivity of Hilbert series on short exact sequences now yields

$$H_{\text{gr}_{\mathfrak{m}}(\Omega_n(M))}(t) = H_{\text{gr}_{\mathfrak{m}}(R)}(t) P_{\Omega_n(M)}^R(-\frac{1}{t}).$$

The rest of the proof follows from that of (1). \square

Setup 2.3. We shall henceforth restrict our attention to the class of local rings (R, \mathfrak{m}) satisfying $\mathfrak{m}^4 = 0$. And, for the sake of simplicity, we will abuse notation in the sequel by allowing

$$H_R(t) = 1 + et + ft^2 + gt^3$$

to represent the Hilbert polynomial of $\text{gr}_{\mathfrak{m}}(R)$. Furthermore, we will often refer to this polynomial as being the Hilbert polynomial of R itself.

Now suppose that M is a finitely generated R -module which admits an eventually linear minimal free resolution and has Betti sequence $\{b_i\}_{i \geq 0}$. If $\text{Ext}_R^i(M, R)$ vanishes for all $i \gg 0$, then Proposition 2.2 yields the following system of equations for each $i, j \gg 0$.

$$(2.3.1) \quad \begin{bmatrix} b_i & -b_{i+1} & b_{i+2} \\ b_{j+3} & -b_{j+2} & b_{j+1} \end{bmatrix} \begin{bmatrix} g \\ f \\ e \end{bmatrix} = \begin{bmatrix} b_{i+3} \\ b_j \end{bmatrix}$$

Row reduction on (2.3.1) yields the reduced system

$$(2.3.2) \quad \begin{bmatrix} b_i & -b_{i+1} & b_{i+2} \\ 0 & \frac{\Delta_1(i, j)}{b_i} & \frac{\Delta_2(i, j)}{b_i} \end{bmatrix} \begin{bmatrix} g \\ f \\ e \end{bmatrix} = \begin{bmatrix} b_{i+3} \\ \frac{\Delta_3(i, j)}{b_i} \end{bmatrix}$$

where

$$(2.3.3) \quad \begin{aligned} \Delta_1(i, j) &= b_{i+1}b_{j+3} - b_ib_{j+2} \\ \Delta_2(i, j) &= b_ib_{j+1} - b_{i+2}b_{j+3} \\ \Delta_3(i, j) &= b_ib_j - b_{i+3}b_{j+3} \end{aligned}$$

for each $i, j \gg 0$.

Given the Betti sequence of such an R -module, one can use Setup 2.3 to determine the explicit Hilbert polynomial of R . However, since (2.3.1) represents infinitely many systems of equations, it might not be immediately obvious that this method is at all well-defined. Nevertheless, Proposition 2.2 insists that, given the Betti sequence $\{b_i\}_{i \geq 0}$ of some R -module, a solution to (2.3.1) will be unique.

The Hilbert polynomial therefore hinges on what possible forms the Betti sequence can take on. This question is discussed in the following section.

3. THE BETTI SEQUENCE

There is very little known about the asymptotic behavior of the Betti sequence of a finitely generated module over an arbitrary local ring. In particular, an answer to the following question of Avramov is still unknown in general.

Question 3.0.1. [1, 4.3.3] Is the Betti sequence of a finitely generated module over a local ring eventually non-decreasing?

A negative answer to this question would introduce the possibility of asymptotic periodicity in the Betti sequence. We consider this question next.

3.1. Periodicity. Our main result for this section will show that Betti sequences associated with linear resolutions over an $\mathfrak{m}^4 = 0$ local ring cannot have ‘small’ periodicity. We make this statement precise in Theorem 3.1.3 below; first, however, we present the following fact.

Fact 3.1.1. *Let M be a finitely generated R -module with Betti sequence $\{b_i\}_{i \geq 0}$. Furthermore, for each $d \in \{1, 2, 3\}$, let $\Delta_d(i, j)$ be the quantity defined in (2.3.3). Then $\Delta_d(i, j) = 0$ for all $i, j \gg 0$ if and only if there exists a positive integer n , which divides d , such that $\{b_i\}_{i \geq 0}$ is eventually periodic of period n .*

Proof. Fix $d \in \{1, 2, 3\}$ and suppose that $b_ib_{j-d+3} = b_{i+d}b_{j+3}$ for all $i, j \gg 0$. Choosing $j = i + d - 3$, one has that $\{b_i\}_{i \geq 0}$ eventually satisfies $b_i^2 = b_{i+d}^2$, which implies that $b_i = b_{i+d}$ for all $i \gg 0$. This implies that $\{b_i\}_{i \geq 0}$ is periodic, and its period clearly divides d .

Conversely, fix $d \in \{1, 2, 3\}$ and suppose that there exists $n \in \mathbb{Z}^+$, with $n \mid d$, such that $\{b_i\}_{i \geq 0}$ is periodic with period n . Then in particular $b_i = b_{i+d}$, and moreover $b_ib_{j-d+3} = b_{i+d}b_{j+3}$, for all $i, j \gg 0$. The result follows. \square

Remarks 3.1.2. (1) The vanishing of $\Delta_d(i, j)$ for all large i and j is essential to obtain periodicity by Fact 3.1.1. Note that even the vanishing of $\Delta_d(i, j)$ for infinitely many pairs (i, j) does not necessarily imply eventual periodicity of $\{b_i\}_{i \geq 0}$.

(2) The condition that $\{b_i\}_{i \geq 0}$ is eventually non-constant is equivalent to the non-vanishing of $\Delta_1(i, j)$ for infinitely many pairs (i, j) .

(3) If $\{b_i\}_{i \geq 0}$ is eventually strictly increasing, then $\Delta_1(i, j) \neq 0$ for all $i, j \gg 0$.

Theorem 3.1.3. *Let M be a finitely generated module with an eventually linear minimal free resolution over a local ring (R, \mathfrak{m}) satisfying $\mathfrak{m}^4 = 0$. If $\text{Ext}_R^i(M, R)$ vanishes for all $i \gg 0$, then the Betti sequence of M is not eventually periodic of period two or three.*

Proof. Assume that $\text{pd}_R M = \infty$, and let the Betti sequence of M be denoted $\{b_i\}_{i \geq 0}$. By Fact 3.1.1, it suffices to show that neither $\Delta_2(i, j)$ nor $\Delta_3(i, j)$, as defined in (2.3.3), can vanish for all $i, j \gg 0$. If the sequence $\{b_i\}_{i \geq 0}$ is eventually non-constant, the set

$$I = \{(n, m) \in \mathbb{N}^2 \mid \Delta_1(n, m) \neq 0\}$$

has infinite cardinality.

First suppose that $\Delta_2(i, j) = 0$ for all $i, j \gg 0$, therefore implying that $\{b_i\}_{i \geq 0}$ eventually has period two. By (2.3.2), one has

$$f = \frac{\Delta_3(n, m)}{\Delta_1(n, m)} = \frac{b_n b_m - b_{n+3} b_{m+3}}{b_{n+1} b_{m+3} - b_n b_{m+2}} = \frac{b_n b_m - b_{n+1} b_{m+1}}{b_{n+1} b_{m+1} - b_n b_m} = -1$$

for all $(n, m) \in I$, which is absurd.

Next suppose that $\Delta_3(i, j) = 0$ for all $i, j \gg 0$, implying that $\{b_i\}_{i \geq 0}$ eventually has period three. By (2.3.2),

$$f = - \left(\frac{\Delta_2(n, m)}{\Delta_1(n, m)} \right) e = \left(\frac{b_{n+2} b_{m+3} - b_n b_{m+1}}{b_{n+1} b_{m+3} - b_n b_{m+2}} \right) e = \left(\frac{b_{n+2} b_m - b_n b_{m+1}}{b_{n+1} b_m - b_n b_{m+2}} \right) e$$

for all $(n, m) \in I$. Notice that since I has infinite cardinality, it is no loss of generality to assume that the set

$$J = \{n \in \mathbb{N} \mid (n, m) \in I \text{ for some } m \in \mathbb{N}\}$$

also has infinite cardinality. Now, for any $n \in J$, if $(n, n) \in I$ then the above equation reduces to $f = -e$, which cannot be true. So it must follow that $(n, n) \notin I$ for any $n \in J$. This implies that

$$\Delta_1(n, n) = b_{n+1} b_{n+3} - b_n b_{n+2} = b_n (b_{n+1} - b_{n+2}) = 0$$

or $b_{n+1} = b_{n+2}$ for all $n \in J$. Because the cardinality of I must be infinite and we assumed that $\{b_i\}_{i \geq 0}$ has period three, it follows that $(n, n+1) = (n, n+2) \in I$.

$$\begin{aligned} (n, n+1) \in I &\implies f = \left(\frac{b_{n+2} b_{n+1} - b_n b_{(n+1)+1}}{b_{n+1} b_{n+1} - b_n b_{(n+1)+2}} \right) e = \left(\frac{b_{n+2}}{b_{n+1} + b_n} \right) e \\ (n, n+2) \in I &\implies f = \left(\frac{b_{n+2} b_{n+2} - b_n b_{(n+2)+1}}{b_{n+1} b_{n+2} - b_n b_{(n+2)+2}} \right) e = \left(\frac{b_{n+2} + b_n}{b_{n+1}} \right) e \end{aligned}$$

These relations hold simultaneously if and only if either $b_n = 0$ or $b_n + b_{n+1} + b_{n+2} = 0$ for all $n \in J$. Since both of these are impossible, we have reached a contradiction. \square

3.2. Growth rates. Among the mystery surrounding the Betti sequence of a finitely generated module over an arbitrary local ring is the following open question of Avramov.

Question 3.2.1. [1, 4.3.7] Does there exist a finitely generated module over a local ring whose Betti sequence grows subexponentially but superpolynomially?

It is known that the Betti sequence of a module over a local ring cannot grow superexponentially by work of Serre [13].

Definition 3.2.2. The Betti sequence $\{b_i\}_{i \geq 0}$ of a finitely generated R -module M is said to have *polynomial growth* if there exists $n \in \mathbb{N}$ such that, for all $i \gg 0$,

$$\alpha i^n - \lambda_i \leq b_i \leq \alpha i^n + \lambda_i$$

for some $\alpha \in \mathbb{R}^+$ and some sequence $\{\lambda_i\}_{i \geq 0}$ of real numbers satisfying $\lambda_i/i^n \rightarrow 0$.

Definition 3.2.3. Let $1 < a \in \mathbb{R}$. The Betti sequence $\{b_i\}_{i \geq 0}$ of a finitely generated R -module M is said to have *exponential growth (of base a)* if, for all $i \gg 0$,

$$\beta a^i - \rho_i \leq b_i \leq \beta a^i + \rho_i$$

for some $\beta \in \mathbb{R}^+$ and some sequence $\{\rho_i\}_{i \geq 0}$ of real numbers satisfying $\rho_i/a^i \rightarrow 0$.

Remarks 3.2.4. (1) The literature often refers to such growth rates in the language of complexity and curvature; cf. [1, 4.2]. Whereas these quantities specify a smallest upper bound for the asymptotic behavior of certain Betti sequences, our definitions above provide a largest lower bound as well.

(2) Both of the above growth rates have been extensively studied. It is well-known that over a complete intersection ring, every finitely generated module has a Betti sequence which grows polynomially [7]. Furthermore, exponential growth of Betti numbers has been demonstrated in a variety of settings, including over Golod rings [14], Cohen-Macaulay rings of small multiplicity [6, 12], and certain $\mathfrak{m}^3 = 0$ local rings [11].

Finally, we define a special type of polynomial growth which will be of importance to our results in the next section.

Definition 3.2.5. The Betti sequence $\{b_i\}_{i \geq 0}$ of a finitely generated R -module M is said to be *exceptional* if

$$b_{i+1} - b_i = b_{i+3} - b_{i+2}$$

for all $i \gg 0$.

Remark 3.2.6. Any Betti sequence which is either constant or exactly linear (that is, $b_{i+1} = b_i + \alpha$ for some $\alpha \in \mathbb{N}$) is exceptional. However, a Betti sequence $\{b_i\}_{i \geq 0}$ such that

$$b_{i+1} - b_i = b_{i+3} - b_{i+2} \neq b_{i+2} - b_{i+1}$$

for all $i \gg 0$ is also exceptional. Though such growth of Betti numbers may seem pathological, it is known to occur over certain codimension two complete intersections by work of Avramov and Buchweitz [2].

4. THE HILBERT POLYNOMIAL

The ultimate goal of this section is to investigate necessary conditions which must be placed on the Hilbert polynomial of an $\mathfrak{m}^4 = 0$ local ring R in order for the ring to admit a finitely generated module M having an eventually linear resolution which is *partially* complete — in other words, satisfying $\text{Ext}_R^i(M, R) = 0 = \text{Ext}_R^i(M^*, R)$ for all $i \gg 0$. However, there is much to be said about the Hilbert polynomial of such a ring in the more general setting. That is, before considering the ‘partially complete’ condition, we first study the existence of R -modules M , with eventually linear resolutions, satisfying the vanishing of $\text{Ext}_R^i(M, R)$ for $i \gg 0$. The following section investigates an even more general scenario: we don’t require the vanishing of $\text{Ext}_R^i(M, R)$ for any i .

4.1. The general form. The recursion relation suggested by Proposition 2.2(1) gives one the ability to express the general form for the Hilbert polynomial of an $\mathfrak{m}^4 = 0$ local ring R which admits a finitely generated module M with an eventually linear minimal free resolution as

$$(4.1.0.1) \quad H_R(t) = 1 + et + ft^2 + \left(\frac{b_{i+1}}{b_i} f - \frac{b_{i+2}}{b_i} e + \frac{b_{i+3}}{b_i} \right) t^3$$

for any $i \gg 0$, where $\{b_i\}_{i \geq 0}$ denotes the Betti sequence of M . If one furthermore assumes that the Betti sequence of M has either polynomial or exponential growth, the following result is obtained.

Lemma 4.1.1. *Let M be a finitely generated module with an eventually linear minimal free resolution over a local ring (R, \mathfrak{m}) satisfying $\mathfrak{m}^4 = 0$.*

(1) *If the Betti sequence of M has polynomial growth, then*

$$H_R(t) = 1 + et + ft^2 + (f - e + 1)t^3.$$

(2) *If the Betti sequence of M has exponential growth of base a , then*

$$H_R(t) = 1 + et + ft^2 + (af - a^2e + a^3)t^3.$$

Proof. Let the Betti sequence of M be denoted by $\{b_i\}_{i \geq 0}$. First we prove (1). By the hypothesis, there exists $n \in \mathbb{N}$ such that, for all $i \gg 0$,

$$\alpha i^n - \lambda_i \leq b_i \leq \alpha i^n + \lambda_i$$

for some $\alpha \in \mathbb{R}^+$ and some sequence $\{\lambda_i\}_{i \geq 0}$ satisfying $\lambda_i/i^n \rightarrow 0$. Given these quantities, one obtains the following bound.

$$\begin{aligned} g &= \frac{b_{i+1}e - b_{i+2}f + b_{i+3}}{b_i} \\ &\leq \frac{(\alpha(i+1)^n + \lambda_{i+1})f - (\alpha(i+2)^n - \lambda_{i+2})e + (\alpha(i+3)^n + \lambda_{i+3})}{\alpha i^n - \lambda_i} \end{aligned}$$

Notice that the quantity on the right-hand side can be made arbitrarily close to $f - e + 1$ as $i \rightarrow \infty$, and one can similarly show that g is bounded from below by a quantity that asymptotically approaches $f - e + 1$. Hence $H_R(t) = 1 + et + ft^2 + (f - e + 1)t^3$, which is what was to be proved.

To show (2), let $1 < a \in \mathbb{R}$ be such that, for all $i \gg 0$,

$$\beta a^i - \rho_i \leq b_i \leq \beta a^i + \rho_i$$

for some $\beta \in \mathbb{R}^+$ and some sequence $\{\rho_i\}_{i \geq 0}$ satisfying $\rho_i/a^i \rightarrow 0$. As in the proof of (1), we proceed to bound g .

$$\begin{aligned} g &= \frac{b_{i+1}e - b_{i+2}f + b_{i+3}}{b_i} \\ &\leq \frac{(\beta a^{i+1} + \rho_{i+1})f - (\beta a^{i+2} - \rho_{i+2})e + (\beta a^{i+3} + \rho_{i+3})}{\beta a^i - \rho_i} \end{aligned}$$

Therefore, g is bounded from above by a quantity that can be made arbitrarily close to $af - a^2e + a^3$ as $i \rightarrow \infty$. The same can be shown for a lower bound of g . Thus, $H_R(t) = 1 + et + ft^2 + (af - a^2e + a^3)t^3$, as claimed. \square

We illustrate the application of the characterizations provided by Lemma 4.1.1 in the following example.

Example 4.1.2. Let $R = k[[w, x, y, z]]/(w^2, wx, x^2, y^2, z^2)$ and consider the R -module $M = R/(w, x)$. One can use an inductive argument to show that the n th map in the minimal free resolution of M over R is represented by the block diagonal matrix

$$\begin{bmatrix} w & x & & & & \\ & & w & x & & \\ & & & & \ddots & \\ & & & & & w & x \end{bmatrix}_{n \times 2n}$$

with respect to the standard bases of R^n and R^{2n} , respectively. Thus, M has a linear minimal free resolution and its Betti sequence has exponential growth of base two. One can now use Lemma 4.1.1(2) to recover the last coefficient of $H_R(t)$.

$$\begin{aligned} H_R(t) &= 1 + 4t + 5t^2 + 2t^3 \\ &= 1 + 4t + 5t^2 + (2 \cdot 5 - 2^2 \cdot 4 + 2^3)t^3 \end{aligned}$$

Notice that the statement of Lemma 4.1.1 does not assume anything about the vanishing of $\text{Ext}_R^i(M, R)$ for $i \gg 0$. In the next section, we shall investigate the additional restrictions on $H_R(t)$ which arise if one makes this assumption.

4.2. (Eventual) vanishing of Ext. The results in this section rely on the system in (2.3.2) having full rank. Since this system reduces to a single equation whenever the Betti sequence is eventually constant, we shall restrict our attention to Betti sequences which are eventually non-constant.

The following proposition specifies conditions under which the Hilbert polynomial of a local ring R can be expressed in terms of the embedding dimension of R and just four Betti numbers of a suitable R -module. This result provides a foundation for the results in the remainder of this manuscript.

Proposition 4.2.1. *Let M be a finitely generated module with an eventually linear minimal free resolution over a local ring (R, \mathfrak{m}) satisfying $\mathfrak{m}^4 = 0$. Suppose that the Betti sequence of M is not eventually constant and that $\text{Ext}_R^i(M, R) = 0$ for all $i > m \geq 0$. If there exists $n \geq \max\{\text{ld}_R(M), m\}$ such that $\Delta_1(n, n) \neq 0$, then*

$H_R(t) = 1 + et + ft^2 + gt^3$, where

$$f = \frac{(b_{n+2}b_{n+3} - b_nb_{n+1})e - (b_{n+3}^2 - b_n^2)}{b_{n+1}b_{n+3} - b_nb_{n+2}}$$

$$g = \frac{(b_{n+2}^2 - b_{n+1}^2)e - (b_{n+2}b_{n+3} - b_nb_{n+1})}{b_{n+1}b_{n+3} - b_nb_{n+2}}$$

given the Betti sequence $\{b_i\}_{i \geq 0}$ of M .

Proof. It suffices to solve the system of equations in (2.3.2). To this end,

$$f = - \left(\frac{\Delta_2(n, n)}{\Delta_1(n, n)} \right) e + \frac{\Delta_3(n, n)}{\Delta_1(n, n)} = \frac{(b_{n+2}b_{n+3} - b_nb_{n+1})e - (b_{n+3}b_{n+3} - b_nb_n)}{b_{n+1}b_{n+3} - b_nb_{n+2}}$$

and

$$g = \left(\frac{b_{n+1}}{b_n} \right) f - \left(\frac{b_{n+2}}{b_n} \right) e + \frac{b_{n+3}}{b_n}.$$

which imply the result upon simplification. \square

Example 4.2.2. Let $R = k[[w, x, y, z]]/(w^2, wx, x^2, y^2, z^2)$ be as in Example 4.1.2, but consider the R -module $M = R/(x)$. Since $\text{Ext}_R^i(M, R)$ does not eventually vanish, one would not expect Proposition 4.2.1 to recover the last two coefficients of $H_R(t)$. Indeed,

$$f = 5 \neq \frac{19}{4} = \frac{(b_3b_4 - b_1b_2)e - (b_4^2 - b_1^2)}{b_2b_4 - b_1b_3}$$

$$g = 2 \neq \frac{3}{2} = \frac{(b_3^2 - b_2^2)e - (b_3b_4 - b_1b_2)}{b_2b_4 - b_1b_3}.$$

Remark 4.2.3. Indeed, it is possible to cook up a non-constant sequence $\{b_i\}_{i \geq 0}$ such that $\Delta_1(n, n) = b_nb_{n+2} - b_{n+1}b_{n+3}$ vanishes for all $n \geq 0$; in particular, such a sequence would be periodic of period four. Although Proposition 4.2.1 could not be used explicitly in the presence of such a Betti sequence, one could derive a similar result by exploiting the non-vanishing of $\Delta_1(m, n)$ for some pair (m, n) . Recall that the existence of such a pair (m, n) is guaranteed by Remark 3.1.2(2). The following lemma demonstrates that $\Delta_1(n, n)$ vanishes for all $n \gg 0$ whenever the Betti sequence grows ‘fast enough.’

Lemma 4.2.4. *If the sequence $\{b_i\}_{i \geq 0}$ has either non-constant polynomial or exponential growth, then $\Delta_1(i, i) = b_ib_{i+2} - b_{i+1}b_{i+3} \neq 0$ for all $i \gg 0$.*

Proof. Suppose that $\{b_i\}_{i \geq 0}$ has polynomial growth. According to Definition 3.2.2, there exists $n \in \mathbb{N}$ such that, for all $i \gg 0$,

$$\alpha i^n - \lambda_i \leq b_i \leq \alpha i^n + \lambda_i$$

for some $\alpha \in \mathbb{R}^+$ and some sequence $\{\lambda_i\}_{i \geq 0}$ satisfying $\lambda_i/i^n \rightarrow 0$. This implies the following bound.

$$\frac{b_i}{b_{i+1}} \leq \frac{\alpha i^n + \lambda_i}{\alpha(i+1)^n - \lambda_{i+1}}$$

One can now see that $\frac{b_i}{b_{i+1}} \rightarrow 0$, and therefore $\frac{b_ib_{i+2}}{b_{i+1}b_{i+3}} \rightarrow 0$ as well. Thus, $b_ib_{i+2} < b_{i+1}b_{i+3}$ for all $i \gg 0$.

A similar proof shows the same result whenever $\{b_i\}_{i \geq 0}$ has exponential growth. \square

We now consider the characterization in Proposition 4.2.1 given prescribed behavior in the Betti sequence. We begin by showing that a symmetric Hilbert polynomial is necessary for non-exceptional polynomially growing Betti numbers.

Theorem 4.2.5. *Let M be a finitely generated module with an eventually linear minimal free resolution over a local ring (R, \mathfrak{m}) satisfying $\mathfrak{m}^4 = 0$, and suppose that the Betti sequence of M has non-exceptional polynomial growth. If $\text{Ext}_R^i(M, R)$ vanishes for all $i \gg 0$, then*

$$H_R(t) = 1 + et + et^2 + t^3.$$

Proof. By virtue of Lemma 4.1.1(1), the Hilbert polynomial of R must take the form

$$H_R(t) = 1 + et + ft^2 + (f - e + 1)t^3.$$

We will use this fact, along with the statement of Proposition 4.2.1, to show that $f = e$, implying the result.

Let $\{b_i\}_{i \geq 0}$ denote the Betti sequence of M . The general form of $H_R(t)$ given in (4.1.0.1) implies that

$$g = \left(\frac{b_{i+1}}{b_i} \right) f - \left(\frac{b_{i+2}}{b_i} \right) e + \frac{b_{i+3}}{b_i} = f - e + 1$$

for $i \gg 0$. Solving for f now yields

$$f = \left(\frac{b_{i+2} - b_i}{b_{i+1} - b_i} \right) e - \frac{b_{i+3} - b_i}{b_{i+1} - b_i}$$

for $i \gg 0$. Since, by Lemma 4.2.4, $\Delta_1(i, i)$ is non-vanishing for all $i \gg 0$, one can now equate the expression for f above with that from Proposition 4.2.1 to obtain the equality

$$\frac{(b_{i+2}b_{i+3} - b_ib_{i+1})e - (b_{i+3}^2 - b_i^2)}{b_{i+1}b_{i+3} - b_ib_{i+2}} = \frac{(b_{i+2} - b_i)e - (b_{i+3} - b_i)}{b_{i+1} - b_i}$$

for all $i \gg 0$. This now implies the equation

$$\begin{aligned} (b_ib_{i+2} - b_{i+2}^2 - b_{i+1}b_{i+3})e - (b_ib_{i+2} - b_{i+2}b_{i+3} - b_{i+1}b_{i+3}) \\ = (b_ib_{i+1} - b_{i+1}^2 - b_{i+2}b_{i+3})e - (b_i^2 - b_ib_{i+1} - b_{i+3}^2) \end{aligned}$$

and therefore

$$(b_{i+2} - b_{i+1})(b_{i+1} + b_{i+2} - b_i - b_{i+3})e = (b_{i+3} - b_i)(b_{i+1} + b_{i+2} - b_i - b_{i+3})$$

for all $i \gg 0$. Now, since we have assumed that $\{b_i\}_{i \geq 0}$ is not exceptional, it follows that $b_{i+1} + b_{i+2} - b_i - b_{i+3}$ does not vanish infinitely often. Thus, there exists some $j \gg 0$ such that

$$e = \frac{b_{j+3} - b_j}{b_{j+2} - b_{j+1}}.$$

Substituting this value of e into the expression for f which is obtained by solving the second equation of (2.3.2), one has the following upon simplification.

$$\begin{aligned} f &= \left(\frac{b_{j+2} - b_j}{b_{j+1} - b_j} \right) \left(\frac{b_{j+3} - b_j}{b_{j+2} - b_{j+1}} \right) - \frac{b_{j+3} - b_1}{b_{j+1} - b_j} \\ &= \frac{b_{j+3} - b_j}{b_{j+2} - b_{j+1}} \\ &= e \end{aligned}$$

This, of course, implies the result. \square

The following two examples illustrate Theorem 4.2.5.

Example 4.2.6. Let $R = k[[x, y, z]]/(x^2, y^2, z^2)$. Then k is a totally reflexive R -module; in particular, $\text{Ext}_R^i(k, R) = 0$ for $i > 0$.

By virtue of the fact that R is a complete intersection ring, the Betti sequence of k has polynomial growth; cf. Remark 3.2.4(2). Even better than this, as R is Koszul, one can explicitly write down the Poincaré series of k .

$$P_k^R(t) = \frac{1}{H_R(-t)} = \frac{1}{(1-t)^3} = \sum_{i \geq 0} \binom{i+2}{2} t^i = \frac{1}{2} \sum_{i \geq 0} (i^2 + 3i + 2) t^i$$

Since the Betti sequence of k has quadratic growth, it is not exceptional. Further, the above resolution must be linear since R is Koszul. Also note that the Hilbert polynomial of R , given by $H_R(t) = 1 + 3t + 3t^2 + t^3$, is symmetric.

Example 4.2.7. Let $R = k[[w, x, y, z]]/(w^2, wx, x^2, y^2, z^2)$, which has an embedded deformation given by $k[[w, x, y, z]]/(w^2, x^2, wx) = S \twoheadrightarrow S/(y^2, z^2) \cong R$. If one defines $M = R/(y, z)$, then one can check that $\text{Ext}_R^i(M, R)$ vanishes for $i > 0$. Furthermore, the minimal free resolution of M over R is given by the following sequence.

$$\cdots \rightarrow R^4 \xrightarrow{\begin{bmatrix} y & 0 & z & 0 \\ 0 & z & 0 & y \\ 0 & 0 & -y & -z \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} y & 0 & z \\ 0 & z & -y \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} y & z \end{bmatrix}} R \rightarrow M \rightarrow 0$$

At this point, the Poincaré series of M might be fairly obvious. However, to be thorough set $P = k[[w, x]]/(w^2, wx, x^2)$ and $Q = k[[y, z]]/(y^2, z^2)$, and notice that $P \otimes_k Q \cong R$ and $P \cong M$ as k -algebras. Therefore, let $\mathbf{F} \twoheadrightarrow k$ be a minimal Q -free resolution of $k \cong Q/(y, z)$. The ranks of the free modules in \mathbf{F} are well-understood since k is the residue field of Q ; we exhibit them in the following Poincaré series.

$$(4.2.7.1) \quad P_Q(t) = \frac{1}{H_{\text{gr}_m(Q)}(-t)} = \frac{1}{(1-t)^2} = \sum_{i \geq 0} (i+1) t^i$$

Since $\text{Tor}_i^k(P, Q) = 0$ for all $i > 0$, $\mathbf{F} \otimes_k P$ is a minimal free resolution of M over R . One can therefore conclude that the Poincaré series of M over R is the same as the one given in (4.2.7.1). In particular, the Betti sequence of M is exceptional. Furthermore, recall that the Hilbert polynomial of R is not symmetric; in fact, one has that $H_R(t) = 1 + 4t + 5t^2 + 2t^3$.

We now consider the characterization of the Hilbert polynomial of an $\mathfrak{m}^4 = 0$ local ring whenever it admits modules with exponentially growing Betti numbers.

Theorem 4.2.8. *Let M be a finitely generated module with an eventually linear minimal free resolution over a local ring (R, \mathfrak{m}) with $\mathfrak{m}^4 = 0$, and suppose that the Betti sequence of M has exponential growth of base a . If $\text{Ext}_R^i(M, R)$ vanishes for all $i \gg 0$, then $H_R(t) = 1 + et + ft^2 + gt^3$, where*

$$f = \left(a + \frac{1}{a}\right)e - \left(a^2 + 1 + \frac{1}{a^2}\right)$$

$$g = e - \left(a + \frac{1}{a}\right).$$

Proof. Let $\{b_i\}_{i \geq 0}$ denote the Betti sequence of M . By assumption, for all $i \gg 0$ one has

$$(4.2.8.1) \quad \beta a^i - \rho_i \leq b_i \leq \beta a^i + \rho_i$$

for some $\beta \in \mathbb{R}^+$ and some sequence $\{\rho_i\}_{i \geq 0}$ of real numbers satisfying $\rho_i/a^i \rightarrow 0$.

Since Lemma 4.2.4 guarantees that $\Delta_1(i, i) \neq 0$ for all $i \gg 0$, we proceed by bounding the expressions for f and g found in Proposition 4.2.1 using the bound in (4.2.8.1). In the interest of space, we omit the tedious details which are analogous to those found in the proof of Lemma 4.1.1(2). Indeed, one obtains the following expressions

$$f = \frac{(a^5 - a)e - (a^6 - 1)}{a^4 - a^2}$$

$$g = \frac{(a^4 - a^2)e - (a^5 - a)}{a^4 - a^2}$$

which simplify to yield the result. \square

An immediate corollary to the previous result is apparent if we consider the fact that f and g must be positive integers.

Corollary 4.2.9. *Let M be a finitely generated module with an eventually linear minimal free resolution over a local ring (R, \mathfrak{m}) with $\mathfrak{m}^4 = 0$, and suppose that $\text{Ext}_R^i(M, R)$ vanishes for all $i \gg 0$. If the Betti sequence of M has exponential growth of base a , then*

$$a = r + s\sqrt{\alpha}$$

for some $r, s \in \mathbb{Q}$ and some $\alpha \in \mathbb{Z}^+$ satisfying $r^2 - \alpha s^2 = 1$.

Proof. By Theorem 4.2.8: (1) a must satisfy some quadratic equation with integer coefficients — thus, one can write $a = r + s\sqrt{\alpha}$ for some $r, s \in \mathbb{Q}$ and $\alpha \in \mathbb{N}$; (2) the sum of a and its reciprocal must be an integer.

If $a \in \mathbb{Q}$, one can write $a = \frac{p}{q}$, where $p, q \in \mathbb{Z}^+$ are relatively prime. Then

$$a + \frac{1}{a} = \frac{p}{q} + \frac{q}{p} = \frac{p^2 + q^2}{pq} \in \mathbb{Z}$$

which implies that $p^2 - npq + q^2 = 0$ for some $n \in \mathbb{Z}$. Solving for p yields

$$p = \frac{nq \pm \sqrt{n^2 q^2 - 4q^2}}{2} = \frac{nq \pm q\sqrt{n^2 - 4}}{2}.$$

In order for this quantity to be an integer, it must be true that $n^2 - 4$ is a perfect square. Since there is no Pythagorean triple of the form $(2, m, n)$, it follows that $n = 2$, which corresponds to the case that $p = q = 1$ — a contradiction.

Since a is irrational, one can assume that $s, \alpha \neq 0$. Furthermore, it is an easy exercise to check that

$$a + \frac{1}{a} = \frac{r(r^2 - \alpha s^2 + 1) + s(r^2 - \alpha s^2 - 1)\sqrt{\alpha}}{r^2 - \alpha s^2}$$

which must be an integer, whence it follows that $r^2 - \alpha s^2 = 1$. \square

In light of Lemma 4.1.1, if one wishes to find an $\mathfrak{m}^4 = 0$ local ring which admits linear resolutions and has a Hilbert polynomial which is *not* balanced, it would be natural to expect the ring to only admit exponentially growing Betti sequences. In fact, Theorem 4.2.8 does not even guarantee that the Hilbert polynomial of such a ring is balanced in the case that the module satisfies the vanishing of Ext condition. The next example illustrates this scenario.

Example 4.2.10. Define local rings $S = k[[x, y, z]]/(x^2 - y^2, x^2 - z^2, xy, xz, yz)$ and $Q = k[[u, v]]/(u^2, uv, v^2)$, with maximal ideals $\mathfrak{m}_S = (x, y, z)$ and $\mathfrak{m}_Q = (u, v)$, respectively. According to [5, Construction 3.1], the local ring

$$R := S \otimes_k Q \cong k[[u, v, w, x, y, z]]/(u^2, uv, v^2, x^2 - y^2, x^2 - z^2, xy, xz, yz)$$

admits non-trivial totally reflexive modules; in particular, $M = R/(x, y, z)$ is one such module.

One would assume that the Betti sequence of M over R would coincide with that of $k \cong S/\mathfrak{m}_S$ over S . To check this, first note that $M \cong Q$ as k -algebras. Further, let $\mathbf{F} \rightarrow k$ be a minimal S -free resolution. Since $\mathrm{Tor}_i^k(S, Q)$ vanishes for $i > 0$, it follows that $\mathbf{F} \otimes_k Q$ is a minimal free resolution of M over R . As S is a Gorenstein ring satisfying $\mathfrak{m}_S^3 = 0$, the Betti sequence of the residue field k over S has exponential growth. Therefore, the same must be true of the Betti sequence of M over R .

Furthermore notice that one can easily check that the Hilbert polynomial

$$H_R(t) = 1 + 5t + 7t^2 + 2t^3$$

of R is clearly not balanced. Also, by using the statement of Theorem 4.2.8, one can recover the base a of the exponential growth of the Betti sequence of M . Indeed,

$$\begin{aligned} a &= \frac{e - g \pm \sqrt{(g - e)^2 - 4}}{2} \\ &= \frac{3 \pm \sqrt{5}}{2} \end{aligned}$$

which implies that $a = \frac{3}{2} + \frac{1}{2}\sqrt{5}$.

Remark 4.2.11. Indeed, one can generalize the previous example. To this end, let $S = k[[x_1, \dots, x_n]]/I$ and $Q = k[[y_1, \dots, y_m]]/J$ where I is generated over $k[[x_1, \dots, x_n]]$ by $x_1^2 - x_j^2$ and $x_i x_j$ for $0 \leq i < j \leq n$, and where J is generated over $k[[y_1, \dots, y_m]]$ by $y_i y_j$ for $1 \leq i \leq j \leq m$. It is clear to see that S is a Gorenstein local ring with Hilbert polynomial $H_S(t) = 1 + nt + t^3$, and that Q is a Cohen-Macaulay local ring with Hilbert polynomial $H_Q(t) = 1 + mt$. Furthermore, since S and Q are Tor-independent k -algebras, the Hilbert polynomial of

$R := S \otimes_k Q$ is given by

$$\begin{aligned} H_R(t) &= H_S(t) \cdot H_Q(t) \\ &= (1 + nt + t^2)(1 + mt) \\ &= 1 + (n + m)t + (1 + nm)t^2 + mt^3 \end{aligned}$$

which is only balanced if $m = 1$ or if $n = 2$.

We now turn our attention to necessary conditions for the existence of R -modules M which satisfy both $\text{Ext}_R^i(M, R) = 0$ and $\text{Ext}_R^i(M^*, R) = 0$ for all $i \gg 0$.

4.3. Asymmetric partially complete resolutions. Our ultimate goal for this section is to investigate necessary conditions for an $\mathfrak{m}^4 = 0$ local ring to admit certain asymmetric (eventually) linear resolutions which are partially complete. However, our actual results are even more general than this: we only require that R admit two modules of differing growth in their Betti sequences.

The idea behind the results in this section is that if a local ring admits modules satisfying the hypotheses of both Theorem 4.2.5 and Theorem 4.2.8, then its Hilbert polynomial must take on both of the respective forms. It is straightforward to see that a study of linear vs. linear asymmetric resolutions will not reveal any additional information about the ring's Hilbert polynomial. Therefore, we restrict our investigation to the remaining two cases. We begin by considering the sort of asymmetric growth of Betti numbers which is apparent in Example ??: polynomial vs. exponential growth.

Theorem 4.3.1. *Let M and N be finitely generated modules, each with an eventually linear minimal free resolution over a local ring (R, \mathfrak{m}) satisfying $\mathfrak{m}^4 = 0$, and suppose that $\text{Ext}_R^i(N, R)$ vanishes for $i \gg 0$. If the Betti sequence of M has polynomial growth, whereas the Betti sequence of N has exponential growth, then*

$$H_R(t) = 1 + et + et^2 + t^3.$$

Proof. By Lemma 4.1.1(1), the Hilbert polynomial of R must be balanced; that is, $H_R(t) = 1 + et + ft^2 + (f - e + 1)t^3$. Furthermore, given the growth of the Betti sequence of N and the fact that $\text{Ext}_R^i(N, R) = 0$ for $i \gg 0$, one can use the characterization of f in Theorem 4.2.8 to obtain

$$\begin{aligned} g &= f - e + 1 \\ &= \left(a + \frac{1}{a}\right)e - \left(a^2 + 1 + \frac{1}{a^2}\right) - e + 1 \\ &= \left(a - 1 + \frac{1}{a}\right)e - \left(a^2 + \frac{1}{a^2}\right). \end{aligned}$$

However, by Theorem 4.2.8 one has $g = e - \left(a + \frac{1}{a}\right)$. Equating these expressions for g yields

$$e = a + 1 + \frac{1}{a}.$$

It is straightforward to check that the Hilbert polynomial of an $\mathfrak{m}^4 = 0$ local ring with this embedding dimension must be symmetric. \square

Remark 4.3.2. In light of Theorem 4.3.1, it is impossible for the ring illustrated in Example 4.2.10 to admit Koszul modules with polynomially growing Betti sequences. In particular, that this implies the ring does not have an exact pair of zero divisors.

Our final result essentially states that asymmetric complete resolutions with exponential vs. exponential growth cannot occur.

Theorem 4.3.3. *Let M and N be finitely generated modules, each with an eventually linear minimal free resolution over a local ring (R, \mathfrak{m}) satisfying $\mathfrak{m}^4 = 0$, and suppose that $\text{Ext}_R^i(M, R) = 0 = \text{Ext}_R^i(N, R)$ for all $i \gg 0$. If the Betti sequences of M and N have exponential growth of bases a and b , respectively, then $a = b$.*

Proof. Suppose the contrary. By Theorem 4.2.8 one has

$$g = e - \left(a + \frac{1}{a}\right) = e - \left(b + \frac{1}{b}\right)$$

which simplifies to yield $ab = 1$. Since both a and b must be larger than one, we have reached a contradiction. \square

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